CERTAIN FORMULAS OF OPERATIONAL CALCULUS OF TWO VARIABLES

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UDC 517.942.82:536.24

Two-dimensional Laplace transforms of certain special functions are presented.

The operational calculus in two variables on the basis of the two-dimensional Laplace transform is utilized extensively in various branches of scientific research [1]. In this connection, the table of two-dimensional Laplace transform formulas have a broad range of applications including the most diverse areas of knowledge. This brief report is a continuation of our paper [2] in which one-dimensional Laplace transform formulas are contained. Presented below is a table of new formulas from the operational calculus in two variables. They are arranged in two columns. On the left are presented the functions f(x, y), while on the right are their two-dimensional Laplace transforms F(p, q), where

$$F(p,\ q)=\int_0^\infty\int_0^\infty f(x,\ y)\exp(-px-qy)\,dxdy$$

(Re p, Re q > 0, if other conditions are not indicated). The notation used is standard in the mathematical literature (see [3-5], for instance).

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[See pages 561, 562, and 563 for tables]

Computation Center, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 44, No. 5, pp. 814-818, May, 1983. Original article submitted April 9, 1982.

 0022-0841/83/4405-0560$07.50

TABLE 1. Two-Dimensional Laplace Transforms of Special Functions

№	$f(x, y)$	$F(p, q)$
1	$x^{\nu} y^{\nu-1} J_{\nu}(2\sqrt[4]{x^2 y})$	$\frac{2^{1-\nu}\Gamma(2\nu)}{p^{\nu+1}q}\exp\left(-\frac{1}{8p^2q}\right)D_{-2\nu}\left(p\sqrt{\frac{q}{2}}\right)$, $\operatorname{Re}\nu > 0$
2	$x^{\nu/2} y^{-\nu/2-1}\left[J_{\nu}(2\sqrt{xy}) - \frac{(xy)^{\nu/2}}{\Gamma(\nu+1)}\right]$	$-\frac{1}{p^{\nu+1}}\ln\left(1+\frac{1}{pq}\right)$, $\operatorname{Re}\nu > -1$
3	$(xy)^{-1/2}J_1(2\sqrt{xy})$	$\ln\left(1+\frac{1}{pq}\right)$
4	$x^{\nu/2-1} y^{\nu/2} I_{\nu}(2\sqrt{xy})$	$\frac{\Gamma(\nu)}{q(pq-1)^{\nu}}$, $pq > 1$, $\operatorname{Re}\nu > 0$
5	$(xy)^{-1/2}I_1(2\sqrt{xy})$	$-\ln\left(1-\frac{1}{pq}\right)$, $pq > 1$
6	$x^{\nu/2} y^{-\nu/2-1}\left[I_{\nu}(2\sqrt{xy}) - \frac{(xy)^{\nu/2}}{\Gamma(\nu+1)}\right]$	$-\frac{1}{p^{\nu+1}}\ln\left(1-\frac{1}{pq}\right)$, $pq > 1$, $\operatorname{Re}\nu > -1$
7	$\left(\frac{x}{y}\right)^{\nu/2} I_{\nu}(2\sqrt{xy})$	$\frac{1}{p^{\nu}(pq-1)}$, $pq > 1$, $\operatorname{Re}\nu > -1$
8	$\frac{1}{xy}[I_0(2\sqrt{xy}) - 1]$	$\operatorname{Li}_2\left(\frac{1}{pq}\right)$, $pq > 1$
9	$\frac{1}{x}\left[\frac{1}{\sqrt{xy}}I_1(2\sqrt{xy}) - 1\right]$	$\frac{1}{q}\left[1+(pq-1)\ln\left(1-\frac{1}{pq}\right)\right]$, $pq > 1$
10	$x^{n/2-1} y^{-n/2-1} \times \left[I_n(2\sqrt{xy}) - \frac{(xy)^{n/2}}{n!}\right]$	$\frac{1}{np^n}\left[(p^n q^n - 1)\ln\left(1-\frac{1}{pq}\right) + p^n q^n \sum_{k=1}^{n}\frac{1}{k(pq)^k}\right]$, $pq > 1$
11	$\frac{1}{xy}[J_0(2\sqrt{xy}) + I_0(2\sqrt{xy}) - 2]$	$\frac{1}{2}\operatorname{Li}_2\left(\frac{1}{p^2q^2}\right)$
12	$J_0(2\sqrt[4]{x^2y}) + I_0(2\sqrt[4]{x^2y})$	$\frac{2}{pq} + \frac{\sqrt{\pi}}{p^2q^{3/2}}\exp\left(\frac{1}{4p^2q}\right)\operatorname{erf}\left(\frac{1}{2p\sqrt{q}}\right)$
13	$\frac{1}{\sqrt{y}}[I_0(2\sqrt[4]{x^2y}) - J_0(2\sqrt[4]{x^2y})]$	$\frac{2\sqrt{\pi}}{pq}\exp\left(\frac{1}{4p^2q}\right)\operatorname{erf}\left(\frac{1}{2p\sqrt{q}}\right)$
14	$J_0(2\sqrt[4]{x^2y})\, I_0(2\sqrt[4]{x^2y})$	$\frac{1}{pq}\exp\left(-\frac{1}{p^2q}\right)$
15	$J_0(2\sqrt[4]{xy})\, I_0(2\sqrt[4]{xy})$	$\frac{1}{pq}\cos\frac{1}{\sqrt{pq}}$
16	$x^{-\nu/2}\mathbf{H}_{\nu}(2\sqrt{xy})$	$\frac{p^{-1/2}q^{-\nu-1/2}}{pq+1}$
17	$x^{-\nu/2}\mathbf{L}_{\nu}(2\sqrt{xy})$	$\frac{p^{-1/2}q^{-\nu-1/2}}{pq-1}$, $pq > 1$

TABLE 1 (continued)

№	$f(x, y)$	$F(p, q)$
18	$\left(\frac{x}{y}\right)^{n/2}[J_n(\pm 2\sqrt{xy}) - i\mathbf{H}_n(\pm 2\sqrt{xy})]$	$(\mp i)^n \frac{p^{-n-1/2}\, q^{-1/2}}{\sqrt{pq} \pm i}$
19	$\left(\frac{x}{y}\right)^{n/2}[J_n(\pm 2i\sqrt{xy}) - i\mathbf{H}_n(\pm 2i\sqrt{xy})]$	$(\pm 1)^n \frac{p^{-n-1/2}\, q^{-1/2}}{\sqrt{pq} \mp 1}$
20	$\left(\frac{x}{y}\right)^{\nu/2}\left[J_\nu(2\sqrt{xy})\cos\frac{\nu\pi}{2} + \mathbf{E}_\nu(2\sqrt{xy})\sin\frac{\nu\pi}{2}\right]$	$\frac{p^{-(\nu+1)/2}\, q^{(\nu-1)/2}}{\sqrt{pq}+1}$
21	$\left(\frac{x}{y}\right)^{n/2}[J_n(\pm 2\sqrt{xy}) + i\mathbf{E}_n(\pm 2\sqrt{xy})]$	$\frac{e^{in\pi/2}\, p^{-(n+1)/2}\, q^{(n-1)/2}}{\sqrt{pq} \pm i}$
22	$\left(\frac{x}{y}\right)^{n/2}[J_n(\pm 2i\sqrt{xy}) - i\mathbf{E}_n(\pm 2i\sqrt{xy})]$	$e^{in\pi/2} \frac{p^{-(n+1)/2}\, q^{(n-1)/2}}{\sqrt{pq} \mp 1}$
23	$\operatorname{ber}(2\sqrt[4]{x^2y})$	$\frac{1}{pq} - \frac{\sqrt{\pi}}{2p^2q^{3/2}} \exp\left(-\frac{1}{4p^2q}\right) \operatorname{erfi}\left(\frac{1}{2p\sqrt{q}}\right)$
24	$\operatorname{ber}(2\sqrt[4]{xy})$	$\frac{1}{pq}\left[1 - \frac{\pi}{4\sqrt{pq}} \mathbf{H}_0\left(\frac{1}{2\sqrt{pq}}\right)\right]$
25	$\frac{1}{xy}[\operatorname{ber}(\sqrt{xy}) - 1]$	$\frac{1}{4}\operatorname{Li}_2\left(-\frac{1}{p^2q^2}\right)$
26	$y^{\nu-1}\operatorname{ber}(2\sqrt[4]{x^2y})$	$\frac{\Gamma(2\nu)}{p(2q)^\nu} \exp\left(-\frac{1}{8p^2q}\right)\left[D_{-2\nu}\left(\frac{i}{p\sqrt{2q}}\right) + D_{-2\nu}\left(-\frac{i}{p\sqrt{2q}}\right)\right]$
27	$\frac{1}{x}\operatorname{bei}(2\sqrt{xy})$	$\frac{1}{q}\operatorname{arctg}\frac{1}{pq}$
28	$\frac{1}{\sqrt{y}}\operatorname{bei}(2\sqrt[4]{x^2y})$	$\frac{\sqrt{\pi}}{p\sqrt{q}} \exp\left(-\frac{1}{4p^2q}\right) \operatorname{erfi}\left(\frac{1}{2p\sqrt{q}}\right)$
29	$y^{\nu-3/2}\operatorname{bei}(2\sqrt[4]{x^2y})$	$-\frac{2^{-\nu+1/2}\, i}{pq^{\nu-1/2}} \exp\left(-\frac{1}{8p^2q}\right) \times \left[D_{-2\nu+1}\left(-\frac{ix}{\sqrt{2}}\right) - D_{-2\nu+1}\left(\frac{ix}{\sqrt{2}}\right)\right]$
30	$\frac{1}{\sqrt{xy}}\operatorname{bei}(2\sqrt[4]{xy})$	$\frac{1}{\sqrt{pq}} H_0\left(\frac{1}{2\sqrt{pq}}\right)$
31	$(xy)^{-1/4}\operatorname{bei}'(2\sqrt[4]{xy})$	$\frac{1}{pq}\left[1 - \frac{\pi}{2}\mathbf{H}_1\left(\frac{1}{2\sqrt{pq}}\right)\right]$
32	$\frac{1}{\sqrt{xy}}\operatorname{bei}'(2\sqrt{xy})$	$\operatorname{arctg}\frac{1}{pq}$
33	$\operatorname{ber}^2(2\sqrt{xy}) + \operatorname{bei}^2(2\sqrt{xy})$	$\frac{1}{pq}\left(1 - \frac{4}{p^2q^2}\right)^{-1/2}, \quad pq > \frac{1}{4}$

TABLE 1 (continued)

№	$f(x, y)$	$F(p, q)$
34	$\operatorname{ber}^2(2\sqrt[4]{xy}) + \operatorname{bei}^2(2\sqrt[4]{xy})$	$\frac{1}{pq}\operatorname{ch}\frac{1}{\sqrt{pq}}$
35	$\operatorname{ber}^2(2\sqrt[4]{xy^2}) + \operatorname{bei}^2(2\sqrt[4]{xy^2})$	$\frac{1}{pq}\exp\frac{1}{pq^2}$
36	$(1+xy)^{-a}\mathrm{B}_{1+xy}(a, n+1)$	$\frac{2^{a+1}n!}{a(a+1)_n}(pq)^{(a-1)/2}[-S_{-a,1-a}(2\sqrt{pq}) +$ $+(-1)^n 2^{2n+2}\ a(a+1)_n(n+1)!\times$ $\times S_{-a-2n-2,1-a}(2\sqrt{pq})]$
37	$(1-xy)^{-a}\mathrm{B}_{1-xy}(a, n+1)$	$\frac{2^{a+1}n!}{a(a+1)_n}e^{-i\pi(a+1)/2}(pq)^{(a-1)/2}\times$ $\times[-S_{-a,1-a}(-2i\sqrt{pq})+(-1)^n 2^{2n+2}a(a+1)_n\times$ $\times(n+1)!\,S_{-a-2n-2,1-a}(-2i\sqrt{pq})]$
38	$P_n^{(0,a-n-1)}(1+2xy)$	$2^{a-n}(pq)^{(a-n-2)/2}S_{n-a-1,n+a}(2\sqrt{pq}),\ a>n$
39	$P_n^{(0,a-n-1)}(1-2xy)$	$2^{a-n}e^{i(n-a)\pi/2}(pq)^{(a-n-2)/2}\times$ $\times S_{n-a-1,n+a}(-2i\sqrt{pq}),\ a>n$
40	$(xy+1)^n P_n^{(-a-n,0)}\left(\frac{1-xy}{1+xy}\right)$	$(-1)^n 2^{a-n}(pq)^{(a-n-2)/2}S_{n-a-1,n+a}(2\sqrt{pq}),$ $a<1-n$
41	$(xy-1)^n P_n^{(-a-n,0)}\left(\frac{xy+1}{xy-1}\right)$	$2^{a-n}e^{i(n-a)\pi/2}(pq)^{(a-n-2)/2}\times$ $\times S_{n-a-1,n+a}(-2i\sqrt{pq}),\quad a<1-n$
42	$P_n(1+2xy)$	$\frac{4}{2n+1}O_{2n+1}(2\sqrt{pq})$
43	$P_n(1-2xy)$	$-\frac{4}{2n+1}O_{2n+1}(-2i\sqrt{pq})$
44	$(1+xy)^{-1/2}P_{2n+1}(\sqrt{1+xy})$	$2^{3/2}(pq)^{-1/4}S_{-1/2,2n+3/2}(2\sqrt{pq})$
45	$(1-xy)^{-1/2}P_{2n+1}(\sqrt{1-xy})$	$2^{3/2}e^{-3\pi i/4}(pq)^{-1/4}S_{-1/2,2n+3/2}(-2i\sqrt{pq})$
46	$(1+xy)^n P_n\left(\frac{1-xy}{1+xy}\right)$	$\frac{1}{2^{2n}(pq)^{2n+1}}S_{2n+1,0}(2\sqrt{pq})$
47	$(1-xy)^n P_n\left(\frac{1+xy}{1-xy}\right)$	$\frac{(-1)^n}{2^{2n}(pq)^{2n+1}}S_{2n+1,0}(-2i\sqrt{pq})$
48	$L_n(xy)$	$\frac{n!}{(pq)^{n+1}}L_n^{-n-1}(-pq)$